Basic definitions & properties

1. Define modular forms
2. Fundamental domain for $SL_2(\mathbb{Z})$
3. Finite-dimensionality of spaces of mod. forms

Upper half plane : $\mathcal{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

$$= \left\{ \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; \omega_2 \neq 0, \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\} \Big/ \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \sim \lambda \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \ \lambda \in \mathbb{C}^\times$$

$SL_2(\mathbb{R})$ acts by

$$SL_2(\mathbb{R}) \times \mathcal{H} \longrightarrow \mathcal{H}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

$-I$ acts trivially : $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm I\}$

functions on $\mathcal{H}$ that transform "nicely" w.r.t. a discrete subgroup

$\Gamma \subset SL_2(\mathbb{R})$. (e.g. $\underbrace{\Gamma_1 = SL_2(\mathbb{Z})}_{\text{full modular group}}$).

$\Gamma_1 \backslash \mathcal{H}$ (moduli space) = parametrises $\left\{\begin{array}{l}\text{isomorphism classes of elliptic curves} \\ \text{over } \mathbb{C}\end{array}\right\}$

$$\tau = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \rightsquigarrow \Lambda_\tau = \overset{\text{lattice}}{\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2} \subset \mathbb{C} \rightsquigarrow \underset{\text{elliptic curve}}{E_\tau = \mathbb{C}/\Lambda_\tau}$$

$\tau \longleftarrow$ choose oriented basis ω_1, ω_2 for Λ $\quad E = \mathbb{C}/\Lambda$

(Silverman: Arith. of ell. curves.)

any two choices are related by $SL_2(\mathbb{Z}) = \Gamma_1$.

What's a <u>modular function</u> on Γ_1? Equivalently:

1. a function $\Gamma_1 \backslash \mathcal{H} \to \mathbb{C}$

2. $f: \mathcal{H} \to \mathbb{C}$ s.t. $f(\gamma z) = f(z)$ $\forall \gamma \in \Gamma_1, \forall z \in \mathcal{H}$

3. a function $\left\{ \text{ell. curves}/\mathbb{C} \right\} / \text{isom} \longrightarrow \mathbb{C}$

4. $F: \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ s.t. $F(\lambda\Lambda) = F(\Lambda)$ $\forall \lambda \in \mathbb{C}^\times, \forall \Lambda$ lattice

Typically require analytic properties:

in ②, $f: \mathcal{H} \to \mathbb{C}$ is meromorphic with (at most) exponential growth at ∞

mod. function = $\frac{\text{holom}}{\text{holom}}$.

modular forms: holom

non-constant

Fix $k \in \mathbb{Z}$ (the weight).

(1) $F : \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ $\quad F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$

$\forall \lambda \in \mathbb{C}^\times, \forall \Lambda$.

(2) $f : \mathcal{H} \to \mathbb{C}$ holomorphic on $\mathcal{H}$

$$f(\gamma z) = (cz+d)^k f(z)$$

$\forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1$

modular form of wt k on Γ_1

with subexponential growth at ∞.

$M_k(\Gamma_1) \overset{\mathbb{C}\text{-vector}}{=} \text{space of}$

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL_2(\mathbb{Z})$, then $f(z+1) = f(z) \quad \forall z \in \mathcal{H}$

Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n q^n \qquad q = e^{2\pi i z}$$

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② Fundamental domain

$-I$ acts trivially on $\mathcal{H}$ $\rightsquigarrow$ consider $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\}$

$$\parallel$$

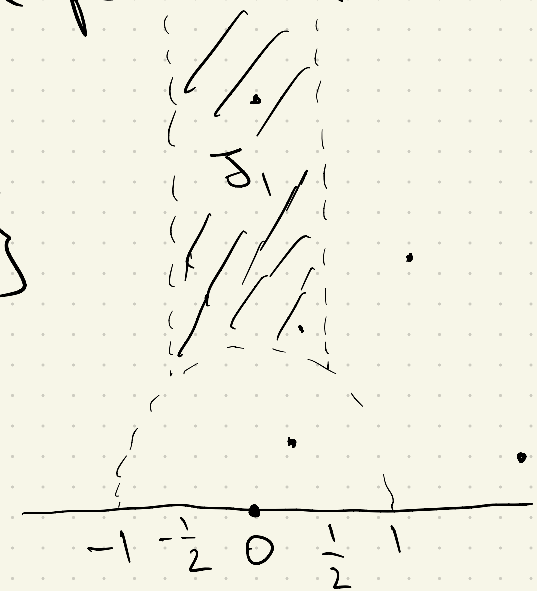
$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow f(z+1) = f(z) \qquad \langle S, T \mid S^2 = (ST)^3 = I \rangle$$

$$f\left(-\frac{1}{z}\right) = z^k f(z) \quad \checkmark$$

$$\Gamma_1 \backslash \mathcal{H}$$

Fundamental domain: $\mathcal{F} \subset \mathcal{H}$ open subset such that no two pts of $\mathcal{F}$ are Γ_1-equivalent and every $z \in \mathcal{H}$ is Γ_1-equivalent to some point in $\overline{\mathcal{F}}$

$\overline{\mathcal{F}}$ = the closure of $\mathcal{F}$.



__Prop__: $\mathcal{F}_1 = \left\{ z \in \mathcal{H} \,\middle|\, |z| > 1 \ \& \ |\mathrm{Re}(z)| < \frac{1}{2} \right\}$ is a fundamental domain for Γ_1

$\dim M_k(\Gamma_1) < \infty$

"identify $M_k(\Gamma_1)$ with the space of global sections of a line bundle on a compact Riemann surface, then clearly finite-dimensional."

- $\Gamma_1 \backslash \mathcal{H}$ is <u>not</u> compact $\rightsquigarrow$ $\Gamma_1 \backslash \overline{\mathcal{H}}$

$\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$

- $\Gamma_1 \backslash \mathcal{H}$ is singular

S (ST)

i, ω

$e^{2\pi i/3}$